2.4 Nonhomogeneous Poisson Process

In this section we generalize the Poisson process by allowing the arrival rate at time t to be a function of t.

Definition 2.4.1

The counting process $\{N(t), t \geq 0\}$ is said to be a nonstationary or nonhomogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$ if

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ has independent increments

(iii) $P\{N(t + h) - N(t) \geq 2\} = o(h)$
(iv) $P\{N(t + h) - N(t) = 1\} = \lambda(t)h + o(h)$

If we let

$$m(t) = \int_0^t \lambda(s)\, ds,$$

then it can be shown that

(2.4 1) $P\{N(t + s) - N(t) = n\}$

$$= \exp\{-(m(t + s) - m(t))\}[m(t + s) - m(t)]^n/n!, \qquad n \geq 0$$

That is, $N(t + s) - N(t)$ is Poisson distributed with mean $m(t + s) - m(t)$.

The proof of (2.4 1) follows along the lines of the proof of Theorem 2.1.1 with a slight modification Fix t and define

$$P_n(s) = P\{N(t + s) - N(t) = n\}$$

Then,

$$\begin{aligned} P_0(s + h) &= P\{N(t + s + h) - N(t) = 0\} \\ &= P\{0 \text{ events in } (t, t + s), 0 \text{ events in } (t + s, t + s + h)\} \\ &= P\{0 \text{ events in } (t, t + s)\}P\{0 \text{ events in } (t + s, t + s + h)\} \\ &= P_0(s)[1 - \lambda(t + s)h + o(h)], \end{aligned}$$

where the next-to-last equality follows from Axiom (ii) and the last from Axioms (iii) and (iv). Hence,

$$\frac{P_0(s + h) - P_0(s)}{h} = -\lambda(t + s)P_0(s) + \frac{o(h)}{h}.$$

Letting $h \to 0$ yields

$$P_0'(s) = -\lambda(t + s)P_0(s)$$

or

$$\log P_0(s) = -\int_0^s \lambda(t + u)\, du$$

or

$$P_0(s) = e^{-[m(t+s)-m(t)]}$$

The remainder of the verification of (2 4.1) follows similarly and is left as an exercise

The importance of the nonhomogeneous Poisson process resides in the fact that we no longer require stationary increments, and so we allow for the possibility that events may be more likely to occur at certain times than at other times

When the intensity function $\lambda(t)$ is bounded, we can think of the nonhomogeneous process as being a random sample from a homogeneous Poisson process. Specifically, let λ be such that

$$\lambda(t) \le \lambda \qquad \text{for all } t \ge 0$$

and consider a Poisson process with rate λ Now if we suppose that an event of the Poisson process that occurs at time t is counted with probability $\lambda(t)/\lambda$, then the process of counted events is a nonhomogeneous Poisson process with intensity function $\lambda(t)$ This last statement easily follows from Definition 2 4 1 For instance (i), (ii), and (iii) follow since they are also true for the homogeneous Poisson process Axiom (iv) follows since

$$\begin{aligned} P\{\text{one counted event in } (t, t+h)\} &= P\{\text{one event in } (t, t+h)\}\frac{\lambda(t)}{\lambda} + o(h) \\ &= \lambda h \frac{\lambda(t)}{\lambda} + o(h) \\ &= \lambda(t)h + o(h) \end{aligned}$$

Also note that, by Proposition 2.3.2 (see the previous supplementary reading), the counted $N(t) \sim \text{Poisson}(\lambda t p)$, where

$$p = \frac{1}{t}\int_0^t P(s)\mathrm{d}s = \frac{1}{t}\int_0^t \frac{\lambda(t)}{\lambda}\mathrm{d}s = \frac{1}{t\lambda}m(t).$$

So, the counted $N(t) \sim \text{Poisson}(m(t))$. To see

the counted $N(t+s)$ $-$ counted $N(t)$ $\sim$ $\text{Poisson}(m(t+s) - m(t))$,

one just needs to prove a general version of Proposition 2.3.2 (where the classification starts from t instead of 0).